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# The joint energy distribution function for the Hamiltonian $H=H_{0}-\mathrm{i} W \boldsymbol{W}^{+}$for the one-channel case 

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Received 9 July 1997


#### Abstract

A closed analytical expression is derived for the joint distribution function of the real and the imaginary parts of the eigenenergies of the operator $H=H_{0}-\mathrm{i} W W^{+}$for the one-channel case, where $H_{0}$ is taken from the Poissonian or one of the Gaussian ensembles with universality index $\beta$, and where the squared moduli $\left|w_{\alpha}\right|^{2}$ of the components of $W$ are assumed to be $\chi^{2}$-distributed with universality index $\bar{\beta}$. In the strong coupling limit and for the special case $\beta=\bar{\beta}$ the joint distribution function of the real parts of the eigenvalues of $H$ becomes identical with the joint energy distribution function of the eigenvalues of $H_{0}$.


## 1. Introduction

All spectroscopic methods face the old problem that the object under observation is unavoidably disturbed by the measurement. Thus, the results obtained from the experiment always reflect an unwanted combination of the properties of the system and the apparatus. Theoretically this can be taken into account by introducing one or several measuring channels by which the system is coupled to the outer world. Quantum mechanically the situation can be described by an effective Hamiltonian

$$
\begin{equation*}
H=H_{0}-\mathrm{i} W W^{+} \tag{1.1}
\end{equation*}
$$

where $H_{0}$ is the Hamiltonian of the 'true' system, undisturbed by the apparatus, and the imaginary part describes the coupling. In the following we shall take for $H_{0}$ an $N \times N$ matrix taken either from the Poissonian or from one of the Gaussian ensembles. In the general case $W$ is given by a $N \times M$ matrix, where $M$ is the number of coupling channels. In the following we shall consider the one-channel case $M=1$ only. Then the matrix $W$ collapses to a vector. For the absolute squares of its components $w_{n}(n=1, \ldots, N)$ we shall assume a $\chi^{2}$ distribution.

The Hamiltonian (1.1) was originally introduced in the context of nuclear physics [Lew91], but it can, in fact, be applied to all situations where the spectroscopic properties are to be determined. In particular it has proved to be useful in the interpretation of the reflection and transmission properties of chaotic microwave cavities ('microwave billiards') [Alt96, Ste95].

In the presence of the channel part $-\mathrm{i} W W^{+}$of the Hamiltonian the eigenvalues acquire a negative imaginary part, and the real parts are modified. Haake, Lehmann and coworkers [Haa88, Leh95] derived analytical expressions for the distributions of eigenvalues of $H$ in the complex plane and found that with increasing coupling strength $g=\sum\left(w_{n}\right)^{2} M$
eigenvalues obtain large negative imaginary parts proportional to $g$, whereas the remaining $N-M$ eigenvalues remain close to the real axis with imaginary parts of the order of $1 / g$. Though qualitatively this result is a straightforward consequence of the special structure of the channel part of the Hamiltonian, it may nevertheless be considered as counterintuitive. The dissipation introduced by the channels is not equally distributed on all eigenvalues, but afflicts essentially only just as many eigenvalues as there are coupling channels.

The model was applied by us to microwave billiards with an attached side arm containing a so-called microwave isolator [Sto95]. It acts as a one-way pass for the microwaves, allowing transmission in one direction only. It showed that these experiments can be quantitatively described by the Hamiltonian (1.1) with one open channel [Haa96].

In the course of the experiments the question arose as to how spectral correlations known from random matrix theory, such as level spacing distribution, number variance or spectral rigidity, have to be modified in the presence of coupling channels. To answer these questions the results in [Leh95] on the eigenvalue distribution in the complex plane do not help as they contain no information on two-point and higher-order correlation functions. For this purpose the correlated eigenenergy distribution function $P\left(E_{n R}, E_{n I}\right)$ of the Hamiltonian, where $E_{n R}$ and $E_{n I}$ are real and imaginary parts of the eigenvalues $E_{n}$ of $H$, respectively is needed. It is the purpose of this paper to derive an expression for this quantity for the one-channel case. Astonishingly enough we shall end with an exact analytical expression. In the derivation only invariance properties will be used, an explicit diagonalization of $H$ will not be necessary.

## 2. Preliminaries

Some preparatory steps are needed, before we start the calculation of the correlated energy distribution function. Most of the results presented in this section are already known from the literature [Alb91]; they are reproduced here mainly for later reference. Starting with the Hamiltonian (1.1) one derives

$$
\begin{equation*}
(E-H)=\left(E-H_{0}\right)\left(1+\frac{\mathrm{i}}{E-H_{0}} W W^{+}\right) . \tag{2.1}
\end{equation*}
$$

Here $E$ is an arbitrary energy (only with the exception of the eigenvalues of $H_{0}$ ). Taking the determinant on both sides one has

$$
\begin{equation*}
|E-H|=\left|E-H_{0}\right|\left|1+\frac{\mathrm{i}}{E-H_{0}} W W^{+}\right| . \tag{2.2}
\end{equation*}
$$

The second determinant can be evaluated with help of the lemma

$$
\begin{equation*}
|1+A B|=|1+B A| . \tag{2.3}
\end{equation*}
$$

For quadratic matrices $A, B$ this is a straightforward consequence of elementary determinant properties, but it also holds for $N \times M$ matrices $A$ and $M \times N$ matrices $B$ with $N \neq M$. For the proof the determinant is written as

$$
|1+A B|=\exp (\operatorname{Tr}(\ln (1+A B)))
$$

Expanding the logarithm and using the commutative property of the trace one immediately arrives at (2.3). Applying the lemma to equation (2.2) one has

$$
\begin{equation*}
|E-H|=\left|E-H_{0}\right|\left|1+\mathrm{i} W^{+} \frac{1}{E-H_{0}} W\right| \tag{2.4}
\end{equation*}
$$

or

$$
\begin{equation*}
\prod_{n}\left(E-E_{n}\right)=\prod_{n}\left(E-E_{n}^{0}\right)\left(1+\mathrm{i} \sum_{\alpha} \frac{\left|w_{\alpha}\right|^{2}}{E-E_{\alpha}^{0}}\right) \tag{2.5}
\end{equation*}
$$

where the $w_{\alpha}$ are the components of $W$ in the eigenbasis of $H_{0}$. One sees that the zeros of the function

$$
\begin{equation*}
g(E)=1+\mathrm{i} \sum_{\alpha} \frac{\left|w_{\alpha}\right|^{2}}{E-E_{\alpha}^{0}} \tag{2.6}
\end{equation*}
$$

give the eigenvalues of $H$ whereas its poles are at the eigenvalues of $H_{0}$. In the strong coupling limit this has the consequences that the eigenvalues of $H$ are pinned down by the two neighbouring poles. The eigenvalues of $H$ and $H_{0}$ thus form an alternating sequence. This alone is sufficient to determine the eigenvalue repulsion behaviour of $H$ for small distances from that of $H_{0}$ [Haa96].

Assuming for the moment $E$ as real, one obtains from equation (2.5) by adding the complex conjugate on both sides,

$$
\begin{equation*}
\prod_{n}\left(E-E_{n}^{0}\right)=\frac{1}{2}\left(\prod_{n}\left(E-E_{n}\right)+\prod_{n}\left(E-E_{n}^{*}\right)\right) \tag{2.7}
\end{equation*}
$$

By subtracting the complex conjugate on both sides of equation (2.5) one obtains analogously

$$
\begin{equation*}
\mathrm{i} \sum_{\alpha} \frac{\left|w_{\alpha}\right|^{2}}{E-E_{\alpha}^{0}}=\frac{\prod\left(E-E_{n}\right)-\prod\left(E-E_{n}^{*}\right)}{\prod\left(E-E_{n}\right)+\prod\left(E-E_{n}^{*}\right)} \tag{2.8}
\end{equation*}
$$

Both equations, though originally derived for real $E$, are nevertheless correct for arbitrary $E$ (the latter one with the exception of the poles, of course). This is a consequence of the principle of analytic continuation. Taking in particular $E=E_{m}$, equation (2.7) yields the relation

$$
\begin{equation*}
\prod_{n}\left(E_{m}-E_{n}^{0}\right)=\frac{1}{2} \prod_{n}\left(E_{m}-E_{n}^{*}\right) \tag{2.9}
\end{equation*}
$$

which will be repeatedly used in the following. For later reference we note that one may equally well interchange the parts of $H$ and $H_{0}$ from the very beginning. One obtains then in complete analogy to equation (2.5)

$$
\begin{equation*}
\prod_{n}\left(E-E_{n}^{0}\right)=\prod_{n}\left(E-E_{n}\right)\left(1-\mathrm{i} W^{+} \frac{1}{E-H} W\right) . \tag{2.10}
\end{equation*}
$$

A combination of equations (2.7) and (2.10) finally yields

$$
\begin{equation*}
\mathrm{i} W^{+} \frac{1}{E-H} W=\frac{1}{2}\left(1-\prod_{n} \frac{\left(E-E_{n}^{*}\right)}{\left(E-E_{n}\right)}\right) \tag{2.11}
\end{equation*}
$$

Now we are prepared to formulate an expression for the correlated energy distribution function $P\left(E_{n R}, E_{n I}\right)$ of the Hamiltonian (1.1). Equation (2.5) supplies us with a relation between the $2 N$ variables $E_{n R}, E_{n I}$ on the left-hand side and the $2 N$ variables $E_{n}^{0}, c_{n}=\left|w_{n}\right|^{2}$ on the right-hand side. The joint energy distribution of $H$ can therefore be obtained from the joint energy distribution of $H_{0}$ and $c_{n}$, and from the Jacobi determinant for the transformation from one set of variables to the other.

If $H_{0}$ belongs to the Poissonian or one of the Gaussian ensembles, the correlated distribution of the $E_{n}^{0}$ is given by [Haa91]

$$
\begin{equation*}
P\left(E_{n}^{0}\right) \sim \prod_{n>m}\left|\left(E_{n}^{0}-E_{m}^{0}\right)\right|^{\beta} \exp \left(-A \sum_{n}\left(E_{n}^{0}\right)^{2}\right) \tag{2.12}
\end{equation*}
$$

where $\beta$ is the universality index. For the $c_{n}$ we assume a $\chi^{2}$-distribution,

$$
\begin{equation*}
p\left(c_{n}\right) \sim\left(\prod_{n} c_{n}\right)^{(\bar{\beta}-2) / 2} \exp \left(-a \sum_{n} c_{n}\right) \tag{2.13}
\end{equation*}
$$

with the universality index $\bar{\beta}$. For $\bar{\beta}=1$ corresponding to the Gaussian orthogonal ensemble (GOE) case this is just the well-known Porter-Thomas distribution. For $\bar{\beta}=0$ corresponding to the Poisson case the distribution is not normalizable. We shall come back to this point later. With these quantities the correlated energy distribution function for $H$ now reads

$$
\begin{equation*}
P\left(E_{n R}, E_{n I}\right) \sim P\left(E_{n}^{0}\right) p\left(c_{n}\right)\left|\frac{\partial\left(E_{n}^{0}, c_{n}\right)}{\partial\left(E_{n R}, E_{n I}\right)}\right| . \tag{2.14}
\end{equation*}
$$

It is now the task to express all quantities entering on the right-hand side of the equation in terms of the $E_{n R}, E_{n I}$ alone. This will be performed step by step in the next section.

## 3. The different steps

### 3.1. The sum $\sum_{n}\left(E_{n}^{0}\right)^{2}$

Expanding the products on both sides of equation (2.7) and equating the coefficients of $E$ and $E^{2}$ one has

$$
\begin{align*}
& \sum_{n} E_{n}^{0}=\frac{1}{2}\left(\sum_{n} E_{n}+\sum_{n} E_{n}^{*}\right) \\
& \sum_{n \neq m} E_{n}^{0} E_{m}^{0}=\frac{1}{2}\left(\sum_{n \neq m} E_{n} E_{m}+\sum_{n \neq m} E_{n}^{*} E_{m}^{*}\right) \tag{3.1}
\end{align*}
$$

whence it follows that, after some straightforward manipulations,

$$
\begin{equation*}
\sum_{n}\left(E_{n}^{0}\right)^{2}=\sum_{n}\left(E_{n R}\right)^{2}-\sum_{n}\left(E_{n I}\right)^{2}+\left(\sum_{n} E_{n I}\right)^{2} . \tag{3.2}
\end{equation*}
$$

### 3.2. The sum $\sum_{n} c_{n}$

Expanding both sides of equation (2.8) in powers of $E^{-1}$ and taking the coefficient of $E^{-1}$ one has

$$
\begin{equation*}
\mathrm{i} \sum_{n} c_{n}=-\frac{1}{2} \sum_{n}\left(E_{n}-E_{n}^{*}\right) \tag{3.3}
\end{equation*}
$$

or

$$
\begin{equation*}
\sum_{n} c_{n}=-\sum_{n} E_{n I} \tag{3.4}
\end{equation*}
$$

### 3.3. The product $\prod_{n} c_{n}$

Taking the limit $E \rightarrow E_{n}^{0}$ on both sides of equation (2.5) one obtains

$$
\begin{equation*}
\mathrm{i} c_{n}=\frac{\prod_{m}\left(E_{n}^{0}-E_{m}\right)}{\prod_{m}^{\prime}\left(E_{n}^{0}-E_{m}^{0}\right)} \tag{3.5}
\end{equation*}
$$

where the prime denotes that the term with $m=n$ has to be omitted from the product. Taking the product over $n$ one has

$$
\begin{equation*}
\prod_{n} c_{n} \sim \frac{\prod_{n m}\left(E_{n}^{0}-E_{m}\right)}{\prod_{n, m}^{\prime}\left(E_{n}^{0}-E_{m}^{0}\right)} \sim \frac{\prod_{n, m}\left(E_{n}-E_{m}^{*}\right)}{\prod_{n, m}^{\prime}\left(E_{n}^{0}-E_{m}^{0}\right)} \tag{3.6}
\end{equation*}
$$

where equation (2.9) was used. The denominator will be treated separately later. We shall see that this quantity will cause the largest troubles.

### 3.4. The Jacobi determinant

The partial logarithmic derivative on both sides of equation (2.5) with respect to $x$, where $x$ stands for one of the $E_{n}^{0}, c_{n}$, yields

$$
\begin{equation*}
-\sum_{m} \frac{1}{E-E_{m}} \frac{\partial E_{m}}{\partial x}=-\sum_{m} \frac{1}{E-E_{m}^{0}} \frac{\partial E_{m}^{0}}{\partial x}+\frac{1}{g(E)} \frac{\partial g(E)}{\partial x} . \tag{3.7}
\end{equation*}
$$

Multiplying both sides with $\left(E-E_{n}\right)$ and taking the limit $E \rightarrow E_{n}$ one obtains

$$
\begin{equation*}
\frac{\partial E_{n}}{\partial x}=-\frac{1}{g^{\prime}\left(E_{n}\right)} \frac{\partial g\left(E_{n}\right)}{\partial x} . \tag{3.8}
\end{equation*}
$$

With

$$
\begin{equation*}
\frac{\partial g\left(E_{n}\right)}{\partial E_{\alpha}^{0}}=\frac{\mathrm{i} c_{\alpha}}{\left(E_{n}-E_{\alpha}^{0}\right)^{2}} \quad \frac{\partial g\left(E_{n}\right)}{\partial c_{\alpha}}=\frac{\mathrm{i}}{E_{n}-E_{\alpha}^{0}} \tag{3.9}
\end{equation*}
$$

it follows for the functional determinant that

$$
J=\left|\frac{\partial\left(E_{n}, E_{n}^{*}\right)}{\partial\left(E_{\alpha}^{0}, c_{\alpha}\right)}\right| \sim \frac{\prod_{n} c_{n}}{\prod_{n} g^{\prime}\left(E_{n}\right)} \left\lvert\, \begin{array}{cc}
\frac{1}{E_{n}-E_{\alpha}^{0}} & \frac{1}{E_{n}^{*}-E_{\alpha}^{0}}  \tag{3.10}\\
\frac{1}{\left(E_{n}-E_{\alpha}^{0}\right)^{2}} & \frac{1}{\left(E_{n}^{*}-E_{\alpha}^{0}\right)^{2}}
\end{array} .\right.
$$

Note that this determinant is the inverse of the quantity entering into equation (2.14). Furthermore we have now taken $E_{n}, E_{n}^{*}$ as variables instead of $E_{n R}, E_{n I}$. The respective Jacobi determinants differ only by a constant factor. For the product $\prod_{n} g^{\prime}\left(E_{n}\right)$ one obtains from equation (2.5)

$$
\begin{align*}
\prod_{n} g^{\prime}\left(E_{n}\right) & =\prod_{n} \frac{\partial}{\partial E}\left(\prod_{m} \frac{\left(E-E_{m}\right)}{\left(E-E_{m}^{0}\right)}\right)_{E=E_{n}} \\
& =\frac{\prod_{n, m}^{\prime}\left(E_{n}-E_{m}\right)}{\prod_{n, m}\left(E_{n}-E_{m}^{0}\right)} \\
& \sim \frac{\prod_{n, m}^{\prime}\left(E_{n}-E_{m}\right)}{\prod_{n, m}\left(E_{n}-E_{m}^{*}\right)} \tag{3.11}
\end{align*}
$$

where in the last step equation (2.9) was again used. The remaining determinant is evaluated in two steps. We start with

$$
\begin{equation*}
\Delta=\left|\frac{1}{E_{n}-E_{\alpha}^{0}}\right| \tag{3.12}
\end{equation*}
$$

This determinant is calculated similarly to the Vandermonde determinant. First the first column is subtracted from all other columns, and the common factors are taken out of the determinant. In the next step the first row is subtracted from all other rows, the common factors are again taken out, and so on. The result is

$$
\begin{equation*}
\Delta=\frac{\prod_{n>m}\left(E_{n}-E_{m}\right) \prod_{\alpha>\beta}\left(E_{\alpha}^{0}-E_{\beta}^{0}\right)}{\prod_{n, \alpha}\left(E_{n}-E_{\alpha}^{0}\right)} \tag{3.13}
\end{equation*}
$$

In the next step the determinant in equation (3.10) is written as

$$
\left.J=\left|\begin{array}{cc}
\frac{1}{E_{n}-E_{\alpha}^{0}} & \frac{1}{E_{n}^{*}-E_{\alpha}^{0}}  \tag{3.14}\\
\frac{1}{\left(E_{n}-E_{\alpha}^{0}\right)^{2}} & \frac{1}{\left(E_{n}^{*}-E_{\alpha}^{0}\right)^{2}}
\end{array}\right|=\prod_{\alpha} \frac{\partial}{\partial \bar{E}_{\alpha}^{0}} \right\rvert\, \begin{array}{cc}
\frac{1}{E_{n}-E_{\alpha}^{0}} & \frac{1}{E_{n}^{*}-E_{\alpha}^{0}} \\
\frac{1}{E_{n}-\bar{E}_{\alpha}^{0}} & \frac{1}{E_{n}^{*}-\bar{E}_{\alpha}^{0}}
\end{array}{\overline{E_{\alpha}^{0}=E_{\alpha}^{0}}} .
$$

The determinant on the right-hand side is exactly of the same type as $\Delta$, with the only difference that now the number of rows and columns has been doubled. One can thus apply the result (3.13) and obtains

$$
\begin{equation*}
D=\frac{\prod_{n>m}\left|E_{n}-E_{m}\right|^{2} \prod_{n, m}\left(E_{n}-E_{m}^{*}\right) \prod_{\alpha>\beta}\left(E_{\alpha}^{0}-E_{\beta}^{0}\right)^{4}}{\prod_{n, \alpha}\left|E_{n}-E_{\alpha}^{0}\right|^{4}} \tag{3.15}
\end{equation*}
$$

Gathering the results one finds for the functional determinant

$$
\begin{equation*}
J=\left|\frac{\partial\left(E_{n}, E_{n}^{*}\right)}{\partial\left(E_{\alpha}^{0}, c_{\alpha}\right)}\right| \sim \prod_{n>m}\left|\frac{E_{n}^{0}-E_{m}^{0}}{E_{n}-E_{m}}\right|^{2} \tag{3.16}
\end{equation*}
$$

### 3.5. The product $\prod^{\prime}\left(E_{n}^{0}-E_{m}^{0}\right)$

The only quantity entering into equation (2.14) which has not yet been expressed in terms of $E_{n R}$ and $E_{n I}$ is

$$
\begin{equation*}
X=\prod_{n>m}\left|E_{n}^{0}-E_{m}^{0}\right| \tag{3.17}
\end{equation*}
$$

In equation (3.13) we have found a relation between $X$ and the determinant $\Delta$, which can be rewritten as

$$
\begin{equation*}
X \sim \frac{\prod_{n, m}\left|E_{n}-E_{m}^{*}\right|}{\prod_{n>m}\left|E_{n}-E_{m}\right|}|\Delta| \tag{3.18}
\end{equation*}
$$

where again relation (2.9) has been used.
We shall now derive an expression for $|\Delta|^{2}$ depending only on $E_{n R}$ and $E_{n I}$. From the definition (3.12) one has, after some elementary transformations,

$$
\begin{align*}
|\Delta|^{2} & =\left|\sum_{\alpha} \frac{1}{E_{n}-E_{\alpha}^{0}} \frac{1}{E_{\alpha}^{0}-E_{m}^{*}}\right| \\
& =\left|\frac{1}{E_{n}-E_{m}^{*}} \sum_{\alpha}\left(\frac{1}{E_{n}-E_{\alpha}^{0}}-\frac{1}{E_{m}^{*}-E_{\alpha}^{0}}\right)\right| \\
& =\left|\frac{X_{n}-X_{m}^{*}}{E_{n}-E_{m}^{*}}\right| \tag{3.19}
\end{align*}
$$

where

$$
\begin{equation*}
X_{n}=\operatorname{Tr} \frac{1}{E_{n}-H_{0}} \tag{3.20}
\end{equation*}
$$

The resolvent entering into the trace can be written as

$$
\begin{equation*}
\frac{1}{E_{n}-H_{0}}=\frac{1}{E_{n}-H^{+}+\mathrm{i} W W^{+}}=\frac{1}{1+\mathrm{i} R_{n} W W^{+}} R_{n} \tag{3.21}
\end{equation*}
$$

where we have introduced

$$
\begin{equation*}
R_{n}=\frac{1}{E_{n}-H^{+}} \tag{3.22}
\end{equation*}
$$

Further transformations yield

$$
\begin{align*}
\frac{1}{E_{n}-H_{0}} & =\left(1-\mathrm{i} R_{n} W W^{+} \frac{1}{1+\mathrm{i} R_{m} W W^{+}}\right) R_{n} \\
& =\left(1-\mathrm{i} R_{n} W \frac{1}{1+\mathrm{i} W^{+} R_{n} W} W^{+}\right) R_{n} \tag{3.23}
\end{align*}
$$

whence follows for the trace

$$
\begin{equation*}
\operatorname{Tr} \frac{1}{E_{n}-H_{0}}=\operatorname{Tr} R_{n}-\mathrm{i} W^{+} R_{n}^{2} W \frac{1}{1+\mathrm{i} W^{+} R_{n} W} . \tag{3.24}
\end{equation*}
$$

The quantities $W^{+} R_{n} W$ and $W^{+} R_{n}^{2} W$ are obtained directly from equation (2.11):

$$
\begin{equation*}
\mathrm{i} W^{+} R_{n} W=-\left(\mathrm{i} W^{+} \frac{1}{E_{n}^{*}-H} W\right)^{*}=-\frac{1}{2} \tag{3.25}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{i} W^{+} R_{n}^{2} W=\frac{\partial}{\partial E}\left(\mathrm{i} W^{+} \frac{1}{E-H} W\right)_{E=E_{n}^{*}}^{*}=-\frac{1}{2} \frac{\prod_{\alpha}^{\prime}\left(E_{n}-E_{\alpha}\right)}{\prod_{\alpha}\left(E_{n}-E_{\alpha}^{*}\right)} . \tag{3.26}
\end{equation*}
$$

We have thus arrived at an expression for $X_{n}$ and thereby also for $|\Delta|^{2}$, depending only on $E_{n R}, E_{n I}$.

## 4. The joint energy distribution function

Collecting the results from the last section, we obtain from equation (2.14) the joint energy distribution function

$$
\begin{align*}
P\left(E_{n R}, E_{n I}\right) \sim & |\Delta|^{\beta-\bar{\beta}} \prod_{n, m}\left|E_{n}-E_{m}^{*}\right|^{\frac{2 \beta-\bar{\beta}-2}{2}} \prod_{n>m}\left|E_{n}-E_{m}\right|^{2-\beta+\bar{\beta}} \\
& \times \exp \left[-A\left(\sum_{n}\left(E_{n R}\right)^{2}-\sum_{m}\left(E_{n I}\right)^{2}+\left(\sum_{m} E_{n I}\right)^{2}\right)-a \sum_{n} E_{n I}\right] \tag{4.1}
\end{align*}
$$

where

$$
\begin{equation*}
|\Delta|^{2}=\left|\frac{X_{n}-X_{m}^{*}}{E_{n}-E_{m}^{*}}\right| \tag{4.2}
\end{equation*}
$$

with

$$
\begin{equation*}
X_{n}=\sum_{\alpha} \frac{1}{E_{n}-E_{\alpha}^{*}}+\frac{\prod_{\alpha}^{\prime}\left(E_{n}-E_{\alpha}\right)}{\prod_{\alpha}\left(E_{n}-E_{\alpha}^{*}\right)} . \tag{4.3}
\end{equation*}
$$

A detailed discussion of equation (4.1) will have to be reserved to further publications. Here only the special case $\beta=\bar{\beta}$ shall be discussed. It is especially simple as here the inconvenient determinant factor is absent. One then has

$$
\begin{align*}
P\left(E_{n R}, E_{n I}\right)= & \prod_{n, m}\left|E_{n}-E_{m}^{*}\right|^{\frac{\beta-2}{2}} \prod_{n>m}\left|E_{n}-E_{m}\right|^{2} \\
& \times \exp \left[-A\left(\sum_{n}\left(E_{n R}\right)^{2}-\sum_{m}\left(E_{n I}\right)^{2}+\left(\sum_{m} E_{n I}\right)^{2}\right)-a \sum_{n} E_{n I}\right] . \tag{4.4}
\end{align*}
$$

This can be further specialized to the case of large coupling strengths $g=\sum c_{n}$. We know that in this case the imaginary parts of all eigenvalues but one vanish for $g \rightarrow \infty$ whereas for one eigenvalue the imaginary part takes an infinitely large negative value (see the discussion


Figure 1. Level spacing distribution and number variance for the real parts of the eigenvalues of matrices $H=H_{0}-\mathrm{i} W W^{+}$. The eigenvalues were obtained by numerical diagonalization of $300 \times 300$ matrices in the strong coupling limit with $g=\sum_{n}\left(w_{n}\right)^{2}=500$. The figure shows the results for the Poissonian (left), the Gaussian orthogonal (middle), and the Gaussian unitary ensemble (right). The solid lines correspond to the random matrix prediction.
in the introduction). In this limit one obtains from equation (4.4) the correlated distribution function of the real parts of the $(N-1)$ eigenvalues remaining close to the real axis:

$$
\begin{equation*}
P\left(E_{n R}\right) \sim \prod_{n>m}\left|E_{n R}-E_{m R}\right|^{\beta} \exp \left(-A\left(\sum_{n}\left(E_{n R}\right)^{2}\right)\right) \tag{4.5}
\end{equation*}
$$

where only the $(N-1)$ remaining eigenvalues are considered in the product and the sum. We have arrived here at the remarkable result that for the case $\beta=\bar{\beta}$ in the strong coupling limit the correlated distribution function for the real parts of the remaining $(N-1)$ eigenvalues is identical to the Gaussian correlated energy distribution function (2.12) for the $N$ eigenvalues of the original Hamilton operator $H_{0}$.

To check this prediction we determined numerically the eigenvalues of random matrices of the form (1.1) with $\beta=\bar{\beta}$ for a coupling constant $g=500$. The test has been performed for the Poissonian, the Gaussian orthogonal, and the Gaussian unitary ensemble by superimposing the spectra of, in total, some 100 matrices of rank 300. For the Poissonian ensemble the distribution (2.13) of the $c_{n}$ had to be truncated to values $c>0.001$ to suppress the nonintegrable singularity at $c=0$. Figure 1 shows the level spacing distribution and number variance for the real parts of the $N-1$ eigenvalues with a small imaginary part.


Figure 2. Level spacing distribution for the real part of eigenvalues of matrices $H=H_{0}-\mathrm{i} W W^{+}$ for the GOE case for different coupling constants $g=0.1(a), 0.3(b), 0.5(c), 0.7(d), 0.9(e)$, $1.1(f), 1.3(g), 1.5(h)$ and $10(i)$. The solid line corresponds to the Wigner GOE prediction.

In all cases the random matrix predictions are verified perfectly well.
In figure 2 the $g$ dependence of the level spacing distribution of the real parts of $H$ is shown for the Gaussian orthogonal ensemble. Again the expected behaviour is found. For small coupling constants the numerical results are still close to the Gaussian orthogonal ensemble curve, move away from it at intermediate values and they reapproach it again at large values.

Corresponding results for the case $\beta \neq \bar{\beta}$ in the strong coupling limit would be highly desirable, as this is the situation realized in microwave billiards with attached unidirectional channels discussed in the introduction [Sto95, Haa96]. For this, however, a compact expression for $|\Delta|^{2}$ in the strong coupling limit is needed.

## Acknowledgments

This publication would not have been possible without out previous work on microwave billiards with broken time-reversal symmetry which was performed in cooperation with F Haake, Essen, M Kuś, Warshawa, and U Kuhl, Marburg, and was supported by the Sonderforschungsbereiche 'Nichtlineare Dynamik' and 'Unordnung und große Fluktuationen' of the Deutsche Forschungsgemeinschaft. F Haake is thanked in addition for numerous inspiring discussions on different stages of the work.

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